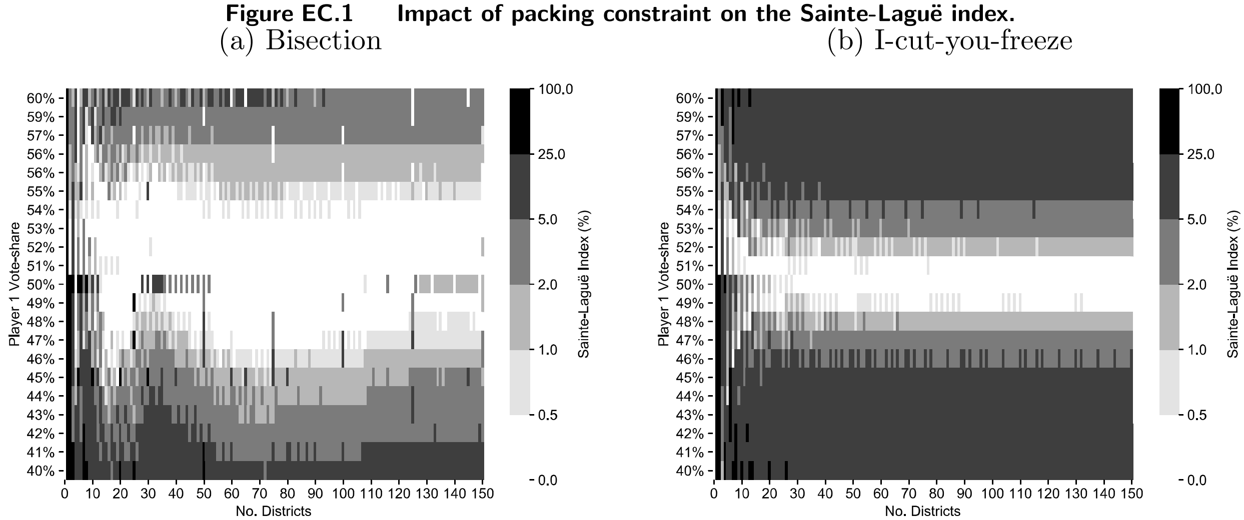
Electronic Companion

Appendix A: Impact of Packing Constraint on Protocol Fairness

This appendix details the effect of a packing constraint (Section 2.3) on the fairness metrics.

Figure EC.1 illustrates the impact of adding the packing constraint with parameter $\delta = 0.25$ on the SL index for both protocols. Recent victory margins (albeit in states with ongoing gerrymandering disputes) suggest $\delta = 0.25$ is large enough to permit districts as extreme as those seen in practice: Maryland's 7th congressional district and Wisconsin's 4th were each won with approximately 75% majorities in 2018 (Maryland State Board of Elections 2018, Wisconsin Elections Commission 2018). A visual comparison of Figures 3 and EC.1 reveals that adding the packing constraint $\delta = 0.25$ to either protocol increases the SL index for nearly all scenarios, meaning the resulting seat-shares are less proportional. This effect, most drastic at normalized vote-shares far from 0.5, accentuates bisection's proportionality advantage over ICYF.

Figure EC.1 Impact of packing constraint on the Sainte-Laguë index.



Note. Plots of the Sainte-Laguë index for 1–150 districts and 40%–60% normalized vote-shares resulting from the (a) bisection and (b) I-cut-you-freeze protocols in the CN setting, under a packing constraint of $\delta = 0.25$.

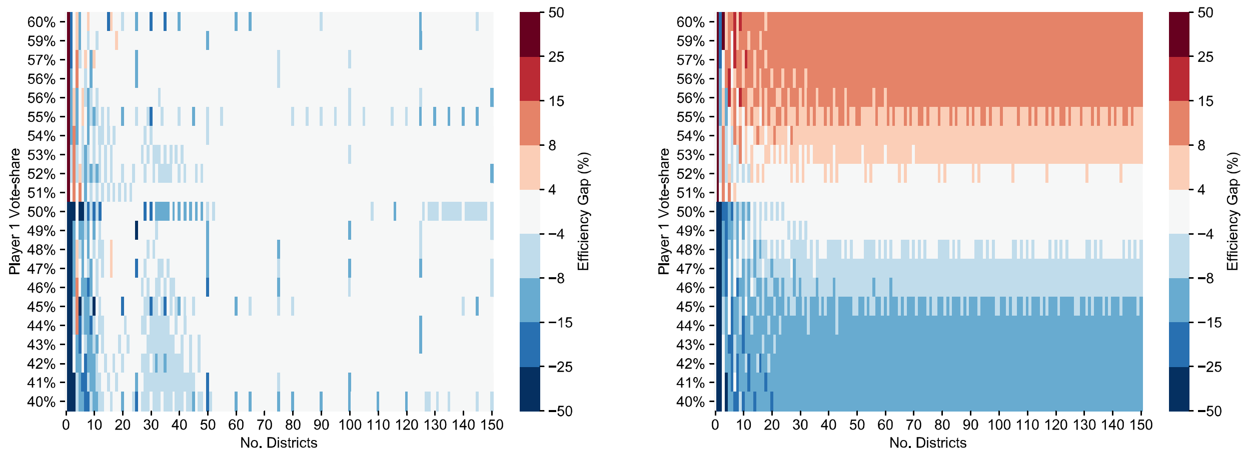
Though inimical to the SL index, the packing constraint improves the efficiency gap in some circumstances. For equipopulous districts and two parties, Stephanopoulos and McGhee (2018) observe the efficiency gap is

$$(\text{seat-margin}) - 2(\text{vote-margin}) = (f(v) - 0.5) - 2(v - 0.5).$$

Hence a protocol utility curve with zero efficiency gap must pass through the point $(0.5, 0.5)$ and have a slope of two.The limiting protocol utility curve for the bisection protocol is this curve exactly when $\delta = 0.25$, meaning the efficiency gap converges to zero for all feasible normalized vote-shares. Figure EC.2 shows the packing constraint's impact on efficiency gap for both protocols. Comparing Figures 4 and EC.2, the packing constraint $\delta = 0.25$ improves the efficiency gap for bisection to within $\pm 8\%$ for most district counts and normalized vote-shares. ICYF, however, produces worse efficiency gaps, generally falling outside $\pm 8\%$ for

Figure EC.2 Impact of packing constraint on efficiency gap.

(a) Bisection (b) I-cut-you-freeze



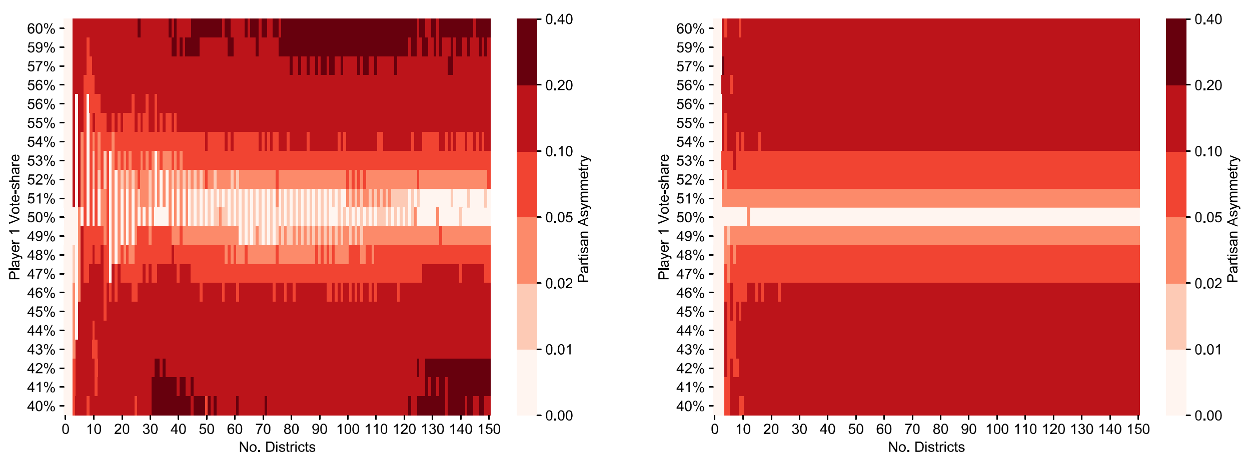
Note. Plots of the efficiency gap for 1–150 districts and 40%–60% normalized vote-shares resulting from the (a) bisection and (b) I-cut-you-freeze protocols in the CN setting, with a $\delta = 0.25$ packing constraint.

normalized vote-shares above 55% and below 45%. One possible explanation is that the weaker player in ICYF heavily relies on strategic packing (see Lemma 3.6 of Pegden et al. 2017).

Partisan asymmetry worsens with the added packing constraint, as evidenced by Figure EC.3 in contrast with Figure 5. Both protocols remain symmetric near 50% normalized vote-share, but asymmetry increases for other vote-shares. Bisection suffers a more pronounced increase than ICYF, especially near the extremes.

Figure EC.3 Impact of packing constraint on partisan asymmetry.

(a) Bisection (b) I-cut-you-freeze



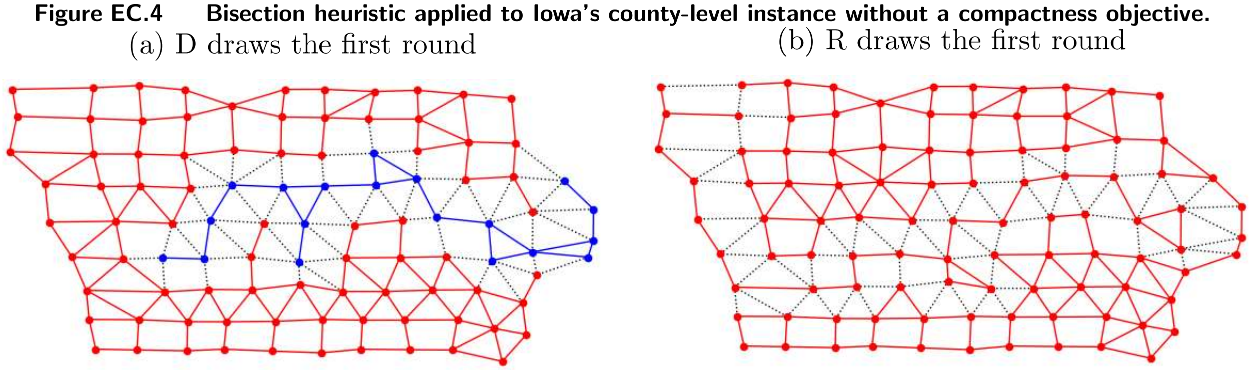
Note. Panels (a) and (b) plot partisan asymmetry for district plans produced by the (a) bisection and (b) ICYF protocols in the CN setting for 1–150 districts and 40%–60% normalized vote-shares, with a $\delta = 0.25$ packing constraint.

Appendix B: Motivation for a Compactness Objective in the Bisection Heuristic

This section motivates the inclusion of a compactness objective in the bisection heuristic (Section 4.1) for the discrete geometric (DG) setting. We show how the bisection heuristic performs with and without a compactness objective when applied to congressional redistricting in Iowa. Recall that in the DG setting, every round of bisection divides each given region into two connected and balanced pieces. The bisection heuristic proposes that the drawing player allocates the same amount of vote-share to each piece as is optimal in the CN setting. Bisecting each round with these vote-share requirements amounts to solving a feasibility problem modeled by MIP constraints (8b)-(8q). If the problem is infeasible, the vote-share requirements are reduced, and the MIP is solved again; this is repeated until a feasible solution is found.

For some instances, solving the feasibility problem results in a bisection into pieces with non-compact shapes. A non-compact bisection in the first round can drastically reduce the number of feasible bisections in subsequent rounds. To illustrate this, consider the first round of bisecting the Iowa instance (Section 4.4); this instance requires two rounds of bisection. The shapes of the bisections produced when either player draws first (depicted in Figure EC.4) are non-compact, snaking around the state. The second round, regardless of the first player, yields no feasible bisection in either of the two pieces. This demonstrates how the first-round bisection may prevent the game from generating any feasible solutions.

Figure EC.4 Bisection heuristic applied to Iowa's county-level instance without a compactness objective.

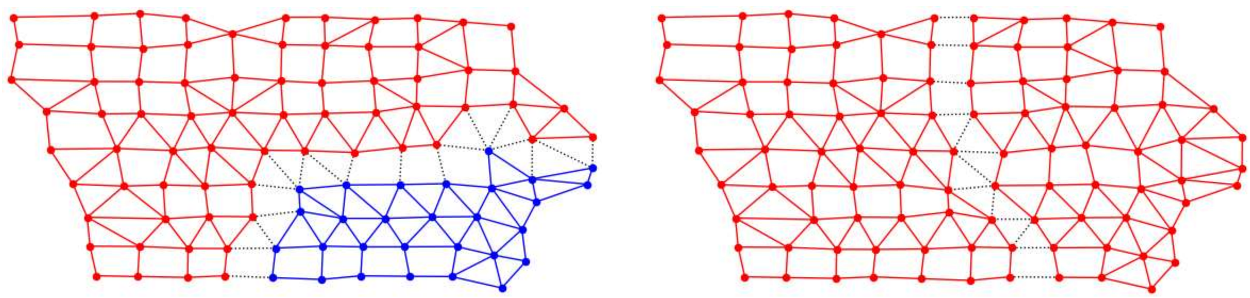


Note. Panels (a) and (b) depict the first round bisection in G when (a) D and (b) R draws the first round. Solid edges are edges within each of the two pieces, while dotted edges are cut-edges.

Consider the bisection MIP with a compactness objective (8a) that minimizes the number of cut-edges (edges that cross two pieces) in the bisection. When the objective is included, the Iowa instance yields a compact first round bisection (depicted in Figure EC.5). The second round, regardless of the first player, yields a feasible bisection in both pieces. See Section 4.4 for an analysis on the second round bisection. Hence, including a compactness objective discourages the bisection heuristic from constructing pieces that would cause the protocol to terminate without producing a feasible district plan.

Figure EC.5 Bisection heuristic applied to Iowa's county-level instance with a compactness objective.

(a) D draws the first round (b) R draws the first round



Note. Panels (a) and (b) depict the first round bisection in G when (a) D and (b) R draws the first round. Solid edges are edges within each of the two pieces, while dotted edges are cut-edges.

Appendix C: Details of Computational Experiments

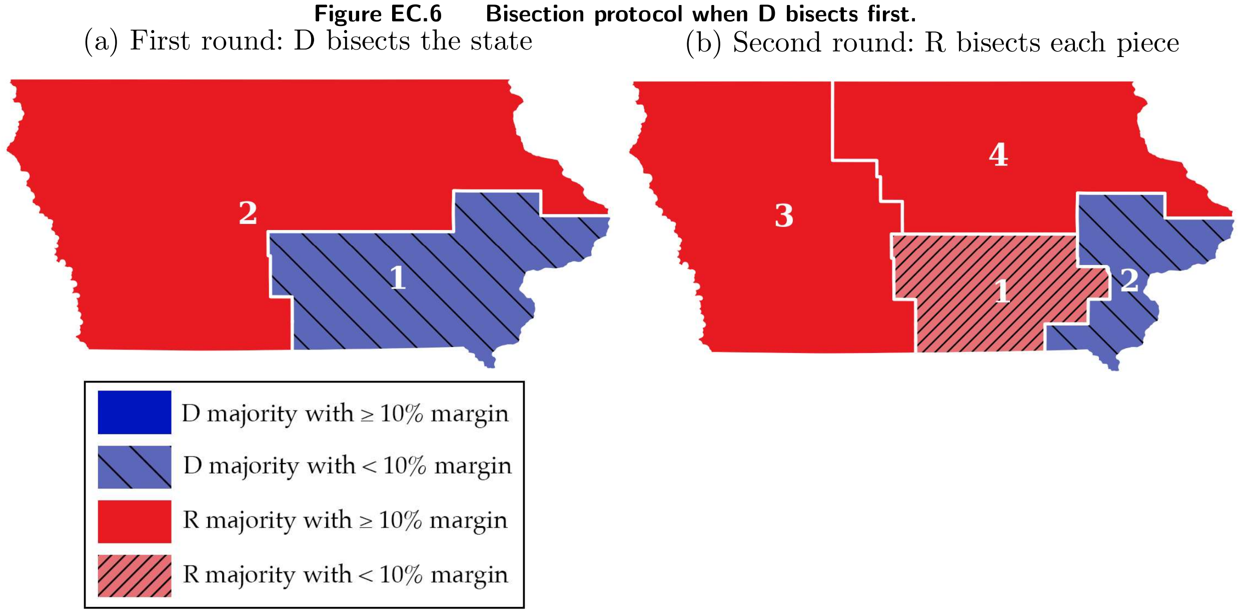
This appendix provides additional details for the computational experiments reported in Sections 4.4 and 4.5.

C.1. Iowa Case Study: Bisection

The bisection protocol (Algorithm 1) is applied to Iowa congressional redistricting using the MIP-based bisection heuristic (Algorithm 3) to divide each set of counties into two pieces (Line 9 of Algorithm 1). With $n = 4$ districts, there are two rounds, that is, two levels of the recursion tree.

Figure EC.6 depicts the results from applying the bisection protocol when D draws the first round and R draws the second round. In the final district plan (Figure EC.6b), D wins one out of the four districts. The vote-shares for D in the four districts are, in decreasing order, 0.54, 0.49, 0.45, and 0.34. This district plan has an SL index of 16.70%, an efficiency gap of 15.49% in favor of R, and a partisan asymmetry of 2.84%. The total CPU time to execute the protocol is 3.26 seconds.

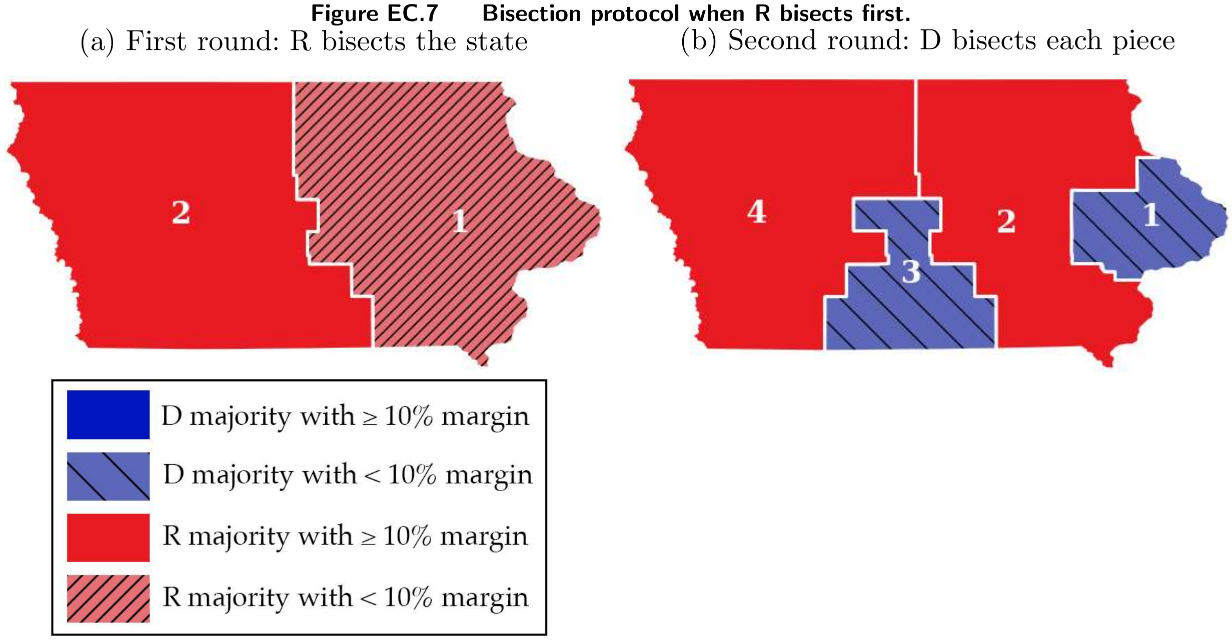
Figure EC.6 Bisection protocol when D bisects first.



When D draws the first round, their normalized vote-share of 45% gives two wins in the CN setting, so they try to win two districts from one piece of the bisection. To achieve this, D first attempts to allocate at least 1.5 vote-share in the first piece, but the MIP is infeasible. Then, D reduces the target vote-share allocation to 1 in the first piece, which yields a feasible bisection. The resulting pieces, labeled 1 and 2 in Figure EC.6a, have D vote-shares 1.02 (51% of the two-district piece) and 0.78 (39% of the two-district piece). Next, R divides the two pieces into four smaller pieces in the next round by allocating 50% (1) and 0% (0) vote shares in each piece's bisection. As illustrated by Figure EC.6b, R first bisects piece 1 into a win for R (district 1) and a win for D (district 2), then bisects piece 2 into two wins for R (districts 3 and 4). Overall, R receives one more seat than the two expected from the CN thresholds.

Swapping turn order, Figure EC.7 depicts the results from applying the bisection protocol when R draws the first round and D draws the second round. In the final district plan (Figure EC.7b), each party wins two districts. The district vote-shares for R are 0.68, 0.58, 0.47, and 0.46. This district plan has an SL Index of 0.88%, an efficiency gap of 9.57% in favor of D, and a partisan asymmetry of 2.27%. The total CPU time to execute the protocol is 2.60 seconds.

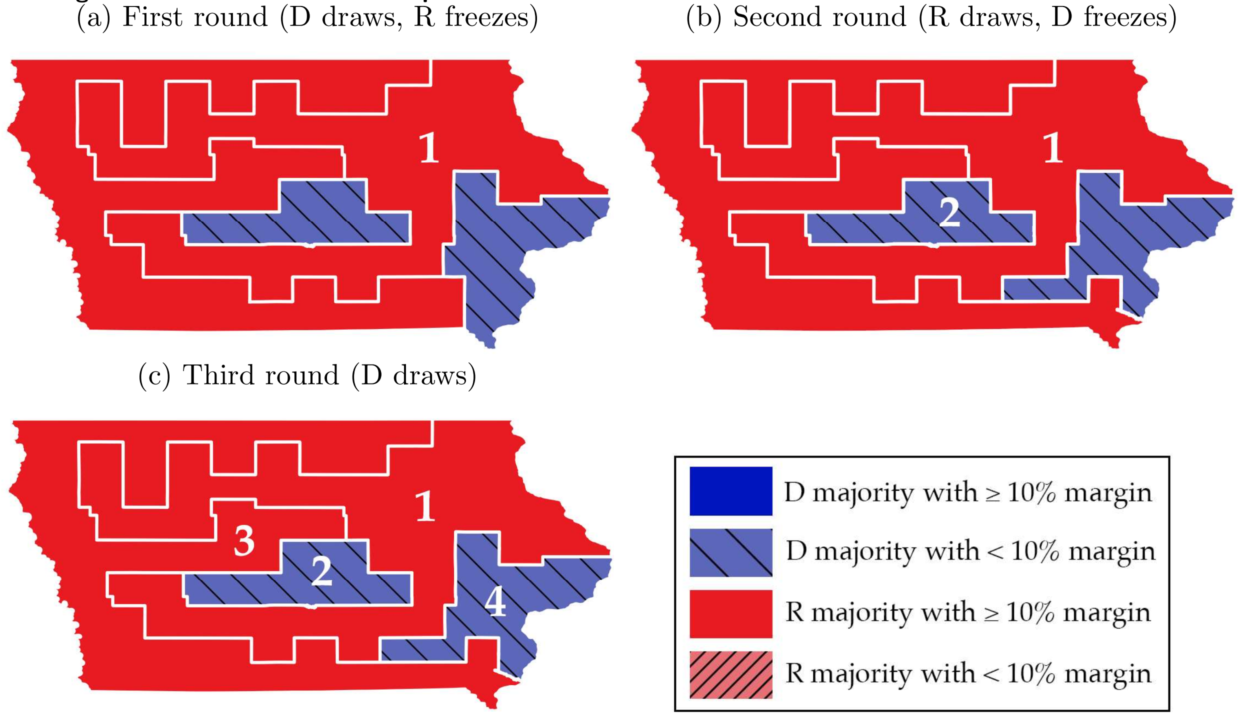
Figure EC.7 Bisection protocol when R bisects first.



When R draws the first round, their normalized vote-share of 55% gives two wins in the CN setting, so they try to win two districts from one piece of the bisection. To achieve this, R first attempts to allocate 1.5 vote-share in the first piece, but the MIP is infeasible. Then, R reduces the target vote-share allocation to 1 in the first piece, which yields a feasible bisection. The resulting pieces, labeled 1 and 2 in Figure EC.7a, have R vote-shares 1.04 (52% of the two-district piece) and 1.14 (57% of the two-district piece). Note these vote-shares do not add to 2.2 since the two pieces have different total numbers of voters. Next, D divides the two pieces into four smaller pieces in the next round by allocating 50% (1) and 0% (0) vote shares in each piece's bisection. As illustrated by Figure EC.7b, D first bisects piece 1 into a win (district 1) and a loss (district 2), then bisects piece 2 into a win (district 3) and a loss (district 4). Overall, R receives two seats, as expected from the CN thresholds.

The DG setting bisection heuristic results differ in strategy and outcome from the outcome expected from the CN setting's thresholds. For the given normalized vote-shares in Iowa, the CN thresholds indicate that the first player (D or R) can win two districts overall, both of which are from one piece of the first bisection. However, when D draws first, the final district plan only yields one D district. When R draws first, the final district plan yields two R districts, although they are spread across the two pieces. This discrepancy between the final outcome and what was expected is due to a combination of (i) the inherent differences between

Figure EC.8 I-Cut-You-Freeze protocol to draw four districts in Iowa when D draws the first round.



the DG and CN settings, and (ii) the approximate nature of applying a heuristic when compared to optimal play.

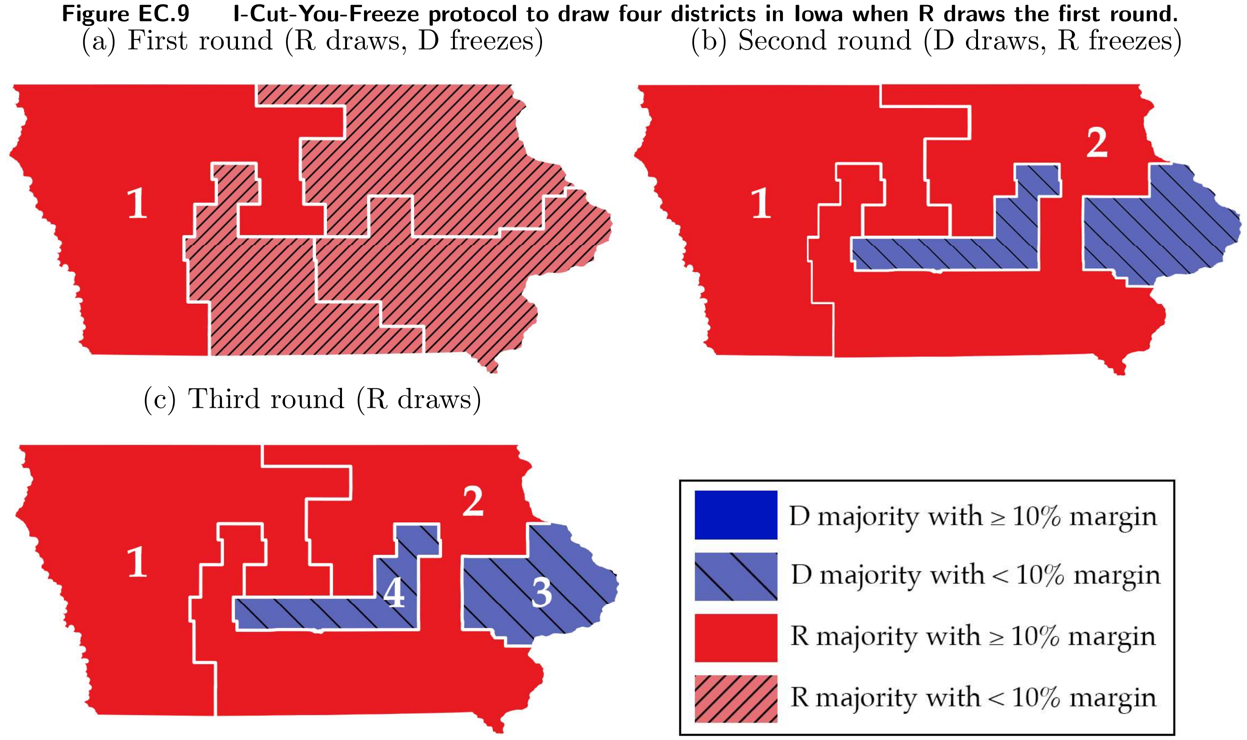
C.2. Iowa Case Study: I-Cut-You-Freeze

The ICYF protocol as described in Algorithm 4 is applied to Iowa congressional redistricting, using the MIP-based ICYF heuristic presented in Section 4.3 to draw the districts in Line 9. For $n = 4$, there are three rounds of ICYF. Note that the third round of drawing essentially freezes the last two districts. In each round, a time limit of 1 hour is set for the MIP solver. If an optimal solution is not found within the time limit, the solver returns the best known solution.

Figure EC.8 depicts heuristic play of the ICYF protocol when D draws first. In the final district plan (Figure EC.8c), D has district vote-shares 0.54, 0.54, 0.41, and 0.33, winning two districts. This district plan has an SL Index of 0.87%, an efficiency gap of 8.39% in favor of D, and a partisan asymmetry of 2.16%. The total computational time to execute the three rounds is 1 hour, 43 minutes.

In the first round, D draws two wins and packs R into two districts with margins 21.51% and 30.93%. Solving the MIP to produce this plan takes 1 hour, when it terminates after reaching the time limit. Following the CN freezing strategy, R freezes district 1, their least-margin win. In the second round, R redraws the remaining three districts, producing the plan shown in Figure EC.8b. Solving this MIP takes 43 minutes. In this plan, R wins one district and packs D into two districts with margins of 7.68% and 8.92%. Then D freezes district 2, their least-margin win. In the third and final round, D draws districts 3 and 4. The MIP solver terminates with optimality in 3 seconds. In the resulting district plan (Figure EC.8c), R wins district 3 and D wins district 4 with margins 31.42% and 8.92%, respectively. As in the CN setting, this plan give two seats to each player.

Figure EC.9 I-Cut-You-Freeze protocol to draw four districts in Iowa when R draws the first round.



Swapping turn order, Figure EC.9 shows heuristic ICYF play when R draws first. In the final district plan (Figure EC.9c), each party wins two districts; the district vote-shares of R are, in decreasing order, 0.64, 0.62, 0.46, and 0.45. This district plan has an SL Index of 0.87%, an efficiency gap of 9.24% in favor of D, and a partisan asymmetry of 0.20%. The total computational time to execute the three rounds is 15 minutes and 35 seconds.

In the first round, R draws four wins and D freezes the win with maximum margin (Figure EC.9a). In the second round, D redraws the remaining three districts, producing the district plan shown in Figure EC.9b. In this plan, D wins two of the three districts and packs R in a district with margin 25.80%, which R freezes to form district 2. The remaining region comprises two disconnected districts, which R is forced to redraw in the third and final round, producing the final district plan (Figure EC.9c). As in the CN setting, this plan give two seats to each player. Note that if R had frozen a D-leaning district in the second round instead of their own packed district, R could have won a third district. Hence the ICYF freezing heuristic is not always optimal.

C.3. Tract-Level Maps for States with 4 to 8 Districts

Figure EC.10 presents the district plans produced by the bisection protocol, using the heuristic of Section 4.1, when D draws first. Figure EC.11 presents the same for when R draws first.

Figure EC.10 District plans obtained from the bisection protocol when D draws the first round.

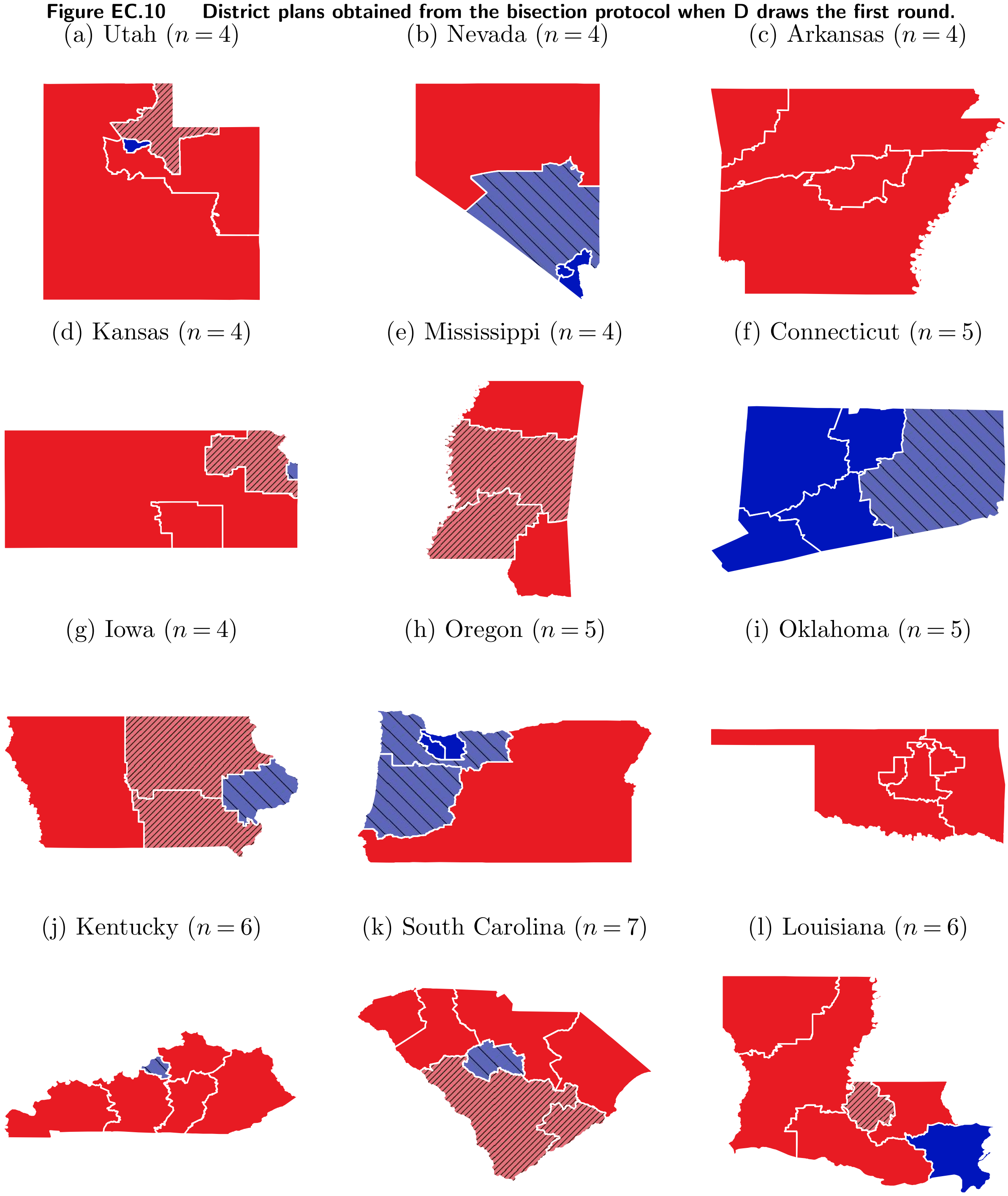


Figure EC.10 (Continued) District plans obtained from the bisection protocol when D draws the first round.

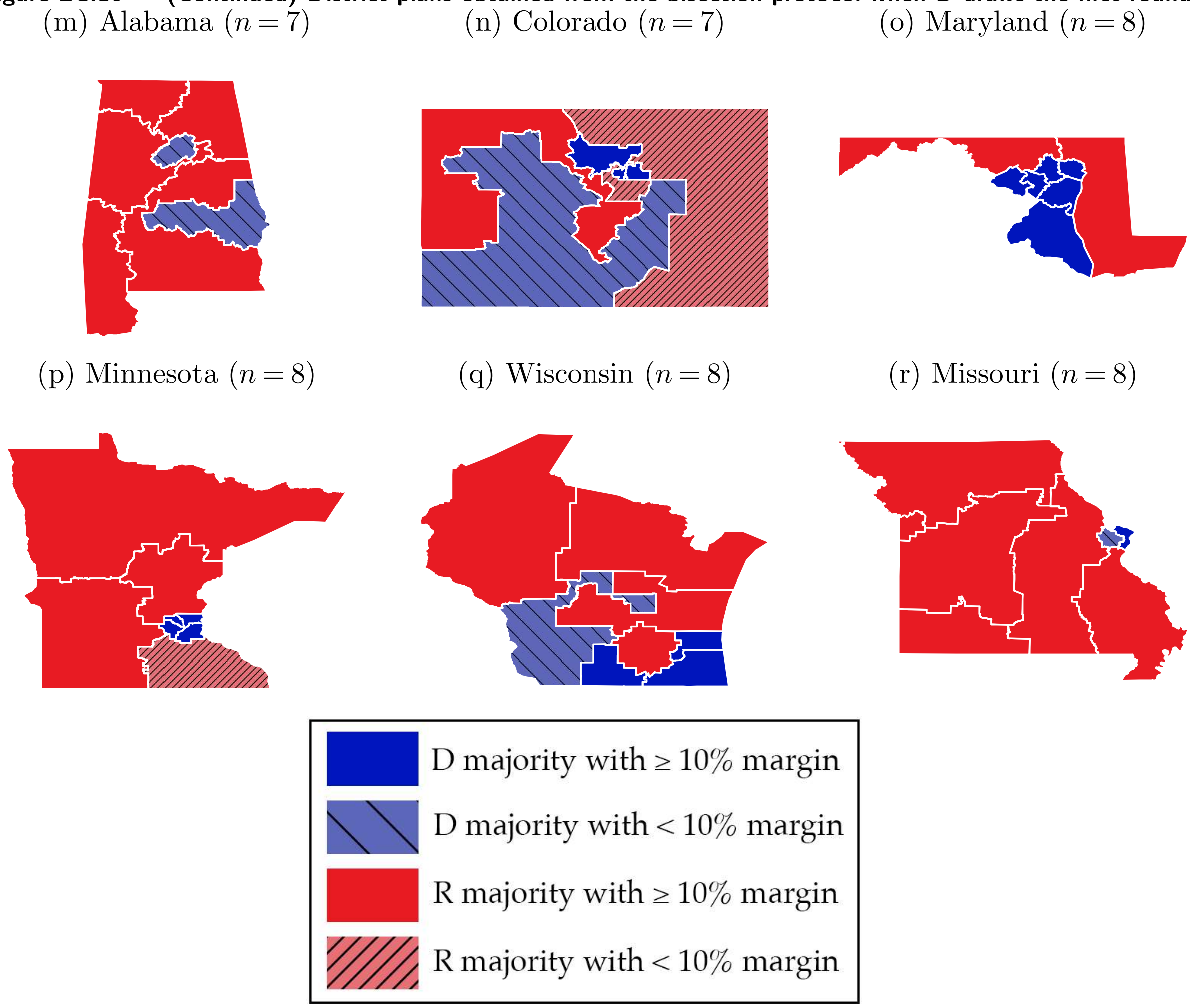


Figure EC.11 District plans obtained from the bisection protocol when R draws the first round.

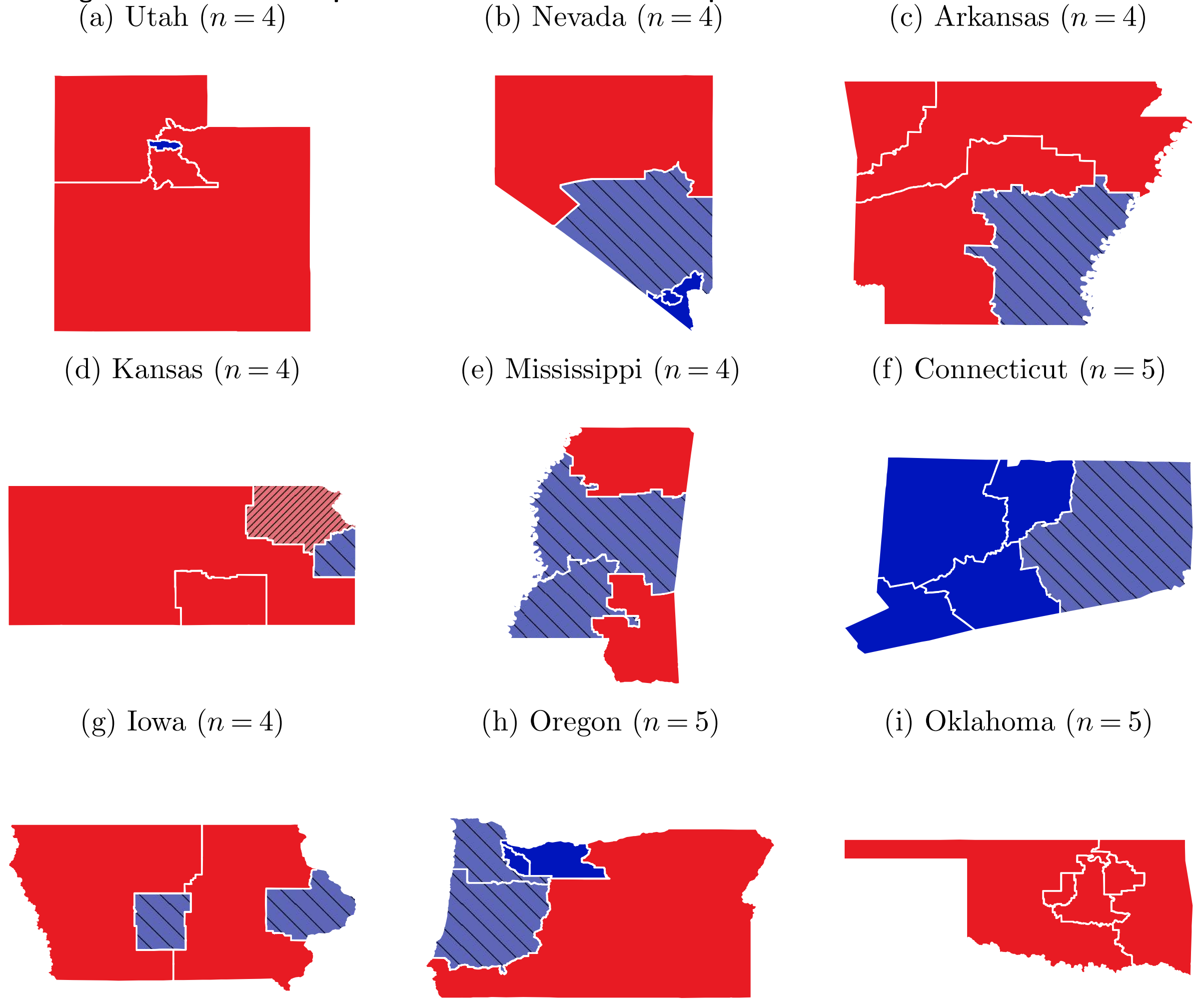
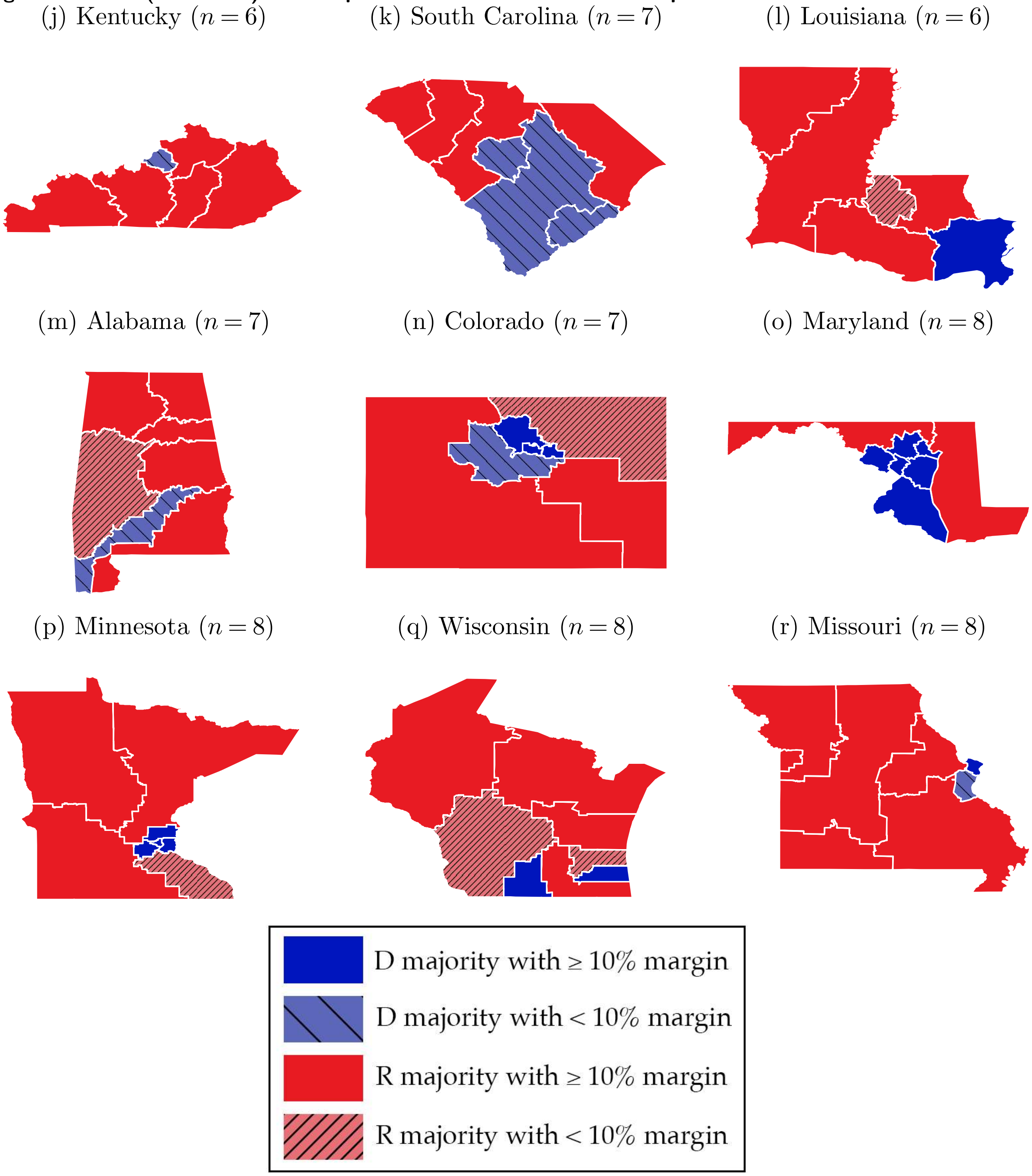


Figure EC.11 (Continued) District plans obtained from the bisection protocol when R draws the first round.



Appendix D: Proofs of Lemmas and Theorems

Throughout the proofs, let $a \mid b$ denote the logical statement "a divides b".

LEMMA 1. *Consider the bisection protocol in the CN setting on n districts. When $n = 2^r$ for some $r \in \mathbb{N}$, the thresholds $t_{n,1}$ and $t_{n,n}$ satisfy the system of recurrences*

$$t_{n,1} = t_{n,n} = 1/2 \text{ for } n = 1,$$
$$t_{n,n} = n - 2t_{n/2,1}, \text{ and}$$
$$t_{n,1} = n/2 - t_{n/2,n/2}.$$

Furthermore, these recurrences have closed-form solutions

$$t_{n,1} = 2^{\lceil (r-1)/2 \rceil - 1} = \begin{cases} \sqrt{n}/2 & \text{if } r \text{ is even} \\ \sqrt{2n}/4 & \text{otherwise} \end{cases}$$

and

$$t_{n,n} = n - 2^{\lceil r/2 \rceil - 1} = \begin{cases} n - \sqrt{n}/2 & \text{if } r \text{ is even} \\ n - \sqrt{2n}/2 & \text{otherwise.} \end{cases}$$

Proof of Lemma 1. First, the recurrences are derived. When $n = 2^0 = 1$, a simple majority of $1/2$ suffices to win the district, so $t_{n,1} = t_{n,n} = 1/2$.

When n is a larger power of two, player 1 wins all districts if and only if player 2 cannot win a single district in either of the two pieces created by player 1 in the first round. This occurs when player 2 has a vote-share of at most $t_{n/2,1}$ in each piece. Hence $t_{n,n} = n - 2t_{n/2,1}$.

Player 1 wins at least one district if and only if player 2 cannot win every district in both pieces. Player 1 can ensure this if and only if player 2 has a vote-share of at most $t_{n/2,n/2}$ in one of the pieces. Player 1 must have enough vote-share to complete that piece, so $t_{n,1} = n/2 - t_{n/2,n/2}$.

The correctness of the closed-form solutions follows from induction.

Base Case: $r = 0$. Then $n = 2^0 = 1$, and $t_{n,1} = 1/2 = \sqrt{n}/2$.

Induction Step: Fix $k > 0$, and suppose the closed-form solutions are correct for all $0 \le r < k$. Let $n = 2^k$. By the induction hypothesis, the closed-form solutions are correct for $n/2$. Substituting the closed-form solution for $t_{n/2,n/2}$ into the recurrence for $t_{n,1}$ yields

$$\begin{aligned} t_{n,1} &= n/2 - n/2 + 2^{\lceil (k-1)/2 \rceil - 1} \\ &= 2^{\lceil (k-1)/2 \rceil - 1}, \end{aligned}$$

the claimed closed-form solution. Similarly, substituting the closed-form solution for $t_{n/2,1}$ into the recurrence for $t_{n,n}$ yields

$$\begin{aligned} t_{n,n} &= n - 2 \cdot 2^{\lceil (k-2)/2 \rceil - 1} \\ &= n - 2^{\lceil k/2 \rceil - 1}, \end{aligned}$$

the claimed closed-form solution. Hence by induction, the closed-form solutions are correct for all $n = 2^r$, $r \in \mathbb{N}$. □

Lemma 2. *Consider the bisection protocol in the CN setting on n districts. Let $j \in \mathbb{N}$, $0 \le j \le n$. If $j = 0$, then $t_{n,j} = 0$. When $n = j = 1$, $t_{n,j} = 1/2$. For $n \ge 2$ and $j \ge 1$,*

$$\begin{aligned} t_{n,j} &= \min_{k \in K(j)} \left\{ (a - t_{a,a-k+1}) + \left(b - t_{b,b-(j-k)+1}\right) \right\} \\ &= \min_{k \in K(j)} \left\{ n - t_{a,a-k+1} - t_{b,b-(j-k)+1} \right\}, \end{aligned}$$

where $a = \lfloor n/2 \rfloor$ is the size of the smaller piece after the cut (side A), $b = \lceil n/2 \rceil$ is the size of the larger piece (side B), and $K(j) = \{k \in \mathbb{N}\colon 0 \le k \le j, 0 \le k \le a, 0 \le j - k \le b\}$ is the set of permissible seat-shares from the part of size a.

Proof of Lemma 2. If $j = 0$, observe that no vote-share is needed to win zero districts, so $t_{n,0} = 0$ for all n. If $n = j = 1$ (a single-district state), then a simple majority wins the district, so $t_{1,1} = 1/2$.

To prove the recurrence, let $n, j \in \mathbb{N}$ with $j \le n$. Suppose without loss of generality that player 1 moves first, cutting the state of size n into pieces A and B with sizes $a = \lfloor n/2 \rfloor$ and $b = \lceil n/2 \rceil$, respectively. To win j districts in total, player 1 must win k districts from A and $j - k$ districts from B for some k such that $0 \le k \le j$, $0 \le k \le a$, and $0 \le j - k \le b$. Let s_a and s_b be player 1's vote-shares in pieces A and B, respectively. Player 1 wins k districts in A if and only if player 2 wins $a - k$ districts in A, which requires $t_{a,a-k} < a - s_a \le t_{a,a-k+1}$. (Recall $t_{a,a+1} = a$ by definition, so the latter inequality holds even when $k = 0$.) Hence $s_a \ge a - t_{a,a-k+1}$. Similarly, player 1 wins $j - k$ districts in B if and only if player 2 wins $b - (j - k)$ districts in B, which requires $t_{b,b-(j-k)} < b - s_b \le t_{b,b-(j-k)+1}$. Hence $s_b \ge b - t_{b,b-(j-k)+1}$. Therefore, an initial vote-share of $s = s_a + s_b \ge (a - t_{a,a-k+1}) + \left(b - t_{b,b-(j-k)+1}\right)$ suffices to win j districts. Taking the minimum such vote-share lower-bound over all feasible k yields the overall minimum for player 1, i.e., the cheapest way to win j districts total between pieces A and B. This minimum is $\min_{k \in K(j)} \left\{n - t_{a,a-k+1} - t_{b,b-(j-k)+1}\right\}$, which is precisely the claimed recurrence for $t_{n,j}$. Hence the recurrence is correct. □

Lemma 3. *Consider the bisection protocol in the CN setting on $n = 2^r$ districts, $r \in \mathbb{N}$. Then $t_{n,0} = 0$, and for all integers $1 \le j \le n$,*

$$t_{n,j} = t_{n,1} + \sum_{i=0}^{\infty} \left(2^{i-2} \left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor \right). \tag{EC.1}$$

Furthermore, an optimal choice of the number of districts to win from one of the two pieces is

$$\operatorname*{arg\,min}_{k \in K(j)} \{n - t_{n/2,n/2-k+1} - t_{n/2,n/2-(j-k)+1}\}, \tag{EC.2}$$

where $K(j)$ is the set of permissible seat-shares from side A as defined in Lemma 2. If $1 \le j \le n/2$, then j is (one of) the minimizer(s) in (2), and if $n/2 < j \le n$, then $n/2$ is the minimizer. These choices form a subgame perfect Nash equilibrium.

The proof of Lemma 3 uses the following facts relating floors and ceilings, which are presented without proof (Graham et al. 1994, Exercise 3.12).

Remark EC.1. For all integers n and all positive integers m,

$$\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n-1}{m} \right\rfloor + 1,$$

and

$$\left\lfloor \frac{n}{m} \right\rfloor = \left\lceil \frac{n+1}{m} \right\rceil - 1.$$

Proof of Lemma 3. The proof is by induction on r. For the base case of $r=0$, $n=2^0=1$, and the right-hand-side sum in (1) is zero for $j=1$. The equation is trivially satisfied as $t_{1,1}=t_{1,1}$. For the base case of $r=1$, $n=2^1=2$, and the right-hand-side sum in (1) is zero for $j=1$. The equation is trivially satisfied as $t_{2,1}=t_{2,1}$. When $r=1$ and $j=2$,

$$\begin{aligned} t_{n,1}+\sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor\frac{2-1}{2^{2i-1}}\right\rfloor\right) &= t_{2,1}+\left(2^{-2}\left\lfloor\frac{1}{1/2}\right\rfloor\right) && \text{(EC.3)}\\ &= 1/2+1/2 \quad \text{(by Lemma 1)} && \text{(EC.4)}\\ &= 1 && \text{(EC.5)}\\ &= 2-2^{\lceil 1/2\rceil-1} && \text{(EC.6)}\\ &= t_{2,2}. \quad \text{(by Lemma 1)} && \text{(EC.7)} \end{aligned}$$

Hence the closed-form solution holds for all valid j when $r=1$.

Let $\ell>1$ be an arbitrary integer, let $n=2^\ell$, and suppose (1) and (2) hold for $r=0,1,\ldots,\ell-1$. Observe that the sum in (1) has a finite number of terms since the floor expression becomes zero when i is sufficiently large.

We first show that (2) holds for $r=\ell$. There are three cases to consider.

Case 1: If $j=n$, then $K(j)=\{n/2\}$ since the only way to win all n districts is to win all $n/2$ from each half. Hence $n/2$ is the argmin in (2) by default.

Case 2: If $n/2<j<n$, then $K(j)=\{j-n/2, j-n/2+1,\ldots,n/2\}$ by definition. To prove $n/2$ is an argmin in (2), it suffices to show for all $k\in K(j)$

$$n-t_{n/2,n/2-n/2+1}-t_{n/2,n/2-(j-n/2)+1}\le n-t_{n/2,n/2-k+1}-t_{n/2,n/2-(j-k)+1},$$

or equivalently,

$$t_{n/2,n/2-k+1}+t_{n/2,n/2-(j-k)+1}\le t_{n/2,1}+t_{n/2,n/2-(j-n/2)+1}. \tag{EC.8}$$

If $k=n/2$, then (EC.8) holds with equality. Suppose $k<n/2$, and let $k'=j-k$ and $j'=j-n/2$. Then the LHS of (EC.8) simplifies to

$$\begin{aligned} &t_{n/2,n/2-k+1}+t_{n/2,n/2-k'+1}\\ &\quad= t_{n/2,1}+\sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor\frac{n/2-k+1-1}{2^{2i-1}}\right\rfloor\right)\\ &\qquad+t_{n/2,1}+\sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor\frac{n/2-k'+1-1}{2^{2i-1}}\right\rfloor\right) && \text{(by I.H.)}\\ &\quad= 2t_{n/2,1}+\sum_{i=0}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{n/2-k}{2^{2i-1}}\right\rfloor\right)\\ &\qquad+\sum_{i=0}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{n/2-k'}{2^{2i-1}}\right\rfloor\right) && \text{(removing terms equal to zero)}\\ &\quad= 2t_{n/2,1}+\frac{n/2-k}{2}+\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left(\left\lceil\frac{n/2-k+1}{2^{2i-1}}\right\rceil-1\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{n/2-k'}{2} + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lceil \frac{n/2-k'+1}{2^{2i-1}}\right\rceil - 1\right)\right) && \text{(by Remark EC.1)} \\
&= 2t_{n/2,1} + n/2 - k/2 - k'/2 - 2\sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\right) \\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\frac{n/2}{2^{2i-1}} - \left\lfloor \frac{k-1}{2^{2i-1}}\right\rfloor\right)\right) \\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\frac{n/2}{2^{2i-1}} - \left\lfloor \frac{k'-1}{2^{2i-1}}\right\rfloor\right)\right) && \text{(since } 2^{2i-1} \mid n/2\text{)} \\
&= 2t_{n/2,1} + n/2 - k/2 - k'/2 - 2\left(\frac{1}{4}\left(2^{\lceil \ell/2\rceil} - 2\right)\right) \\
&\quad + 2\cdot\frac{n}{2}\sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{-i-1}\right) \\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lfloor \frac{k-1}{2^{2i-1}}\right\rfloor\right)\right) \\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lfloor \frac{k'-1}{2^{2i-1}}\right\rfloor\right)\right) && \text{(separating summation terms)} \\
&= 2t_{n/2,1} + n/2 \\
&\quad - 2^{\lceil \ell/2\rceil - 1} + n\left(1/2 - 2^{-\lceil \ell/2\rceil}\right) \\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lfloor \frac{k-1}{2^{2i-1}}\right\rfloor\right)\right) - (k-1)/2 \\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lfloor \frac{k'-1}{2^{2i-1}}\right\rfloor\right)\right) - (k'-1)/2 \\
&= 2t_{n/2,1} + n/2 - 2^{\lceil \ell/2\rceil - 1} + n/2 - n\cdot 2^{-\lceil \ell/2\rceil} \\
&\quad - (t_{n/2,k} - t_{n/2,1}) - (t_{n/2,k'} - t_{n/2,1}) && \text{(by I.H.)} \\
&= 4t_{n/2,1} + n - 2^{\lceil \ell/2\rceil - 1} - n\cdot 2^{-\lceil \ell/2\rceil} \\
&\quad - t_{n/2,k} - t_{n/2,k'} \\
&= 2t_{n/2,1} + n - n\cdot 2^{-\lceil \ell/2\rceil} && \text{(since } 2^{\lceil \ell/2\rceil - 1} = 2t_{n/2,1}\text{)} \\
&\quad - t_{n/2,k} - t_{n/2,k'} \\
&= 2t_{n/2,1} + n - 2^{\lceil(\ell+1)/2\rceil - 1} - t_{n/2,k} - t_{n/2,k'}. && \text{(EC.9)}
\end{aligned}
$$

Similarly, the RHS of (EC.8) simplifies to

$$
\begin{aligned}
&t_{n/2,1} + t_{n/2,n/2-j'+1} \\
&\quad = t_{n/2,1} + t_{n/2,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{n/2-j'+1-1}{2^{2i-1}}\right\rfloor\right) && \text{(by I.H.)} \\
&\quad = 2t_{n/2,1} + \sum_{i=0}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor \frac{n/2-j'}{2^{2i-1}}\right\rfloor\right) \\
&\quad = 2t_{n/2,1} + \frac{n/2-j'}{2} + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor \frac{n/2-j'}{2^{2i-1}}\right\rfloor\right) && \text{(removing terms equal to zero)}
\end{aligned}
$$

$$
\begin{aligned}
&= 2t_{n/2,1} + \frac{n/2 - j'}{2} \\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\frac{n/2}{2^{2i-1}} - \left\lceil \frac{j'}{2^{2i-1}} \right\rceil\right)\right) && \text{(since } 2^{2i-1} \mid n/2) \\
&= 2t_{n/2,1} + \frac{n}{4} - \frac{j'}{2} + \frac{n}{2} \cdot \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{-i-1}\right) \\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lfloor \frac{j'-1}{2^{2i-1}} \right\rfloor + 1\right)\right) && \text{(by Remark EC.1)} \\
&= 2t_{n/2,1} + \frac{n}{4} + \frac{n}{2} \cdot \left(\frac{1}{2} - 2^{-\lceil \ell/2 \rceil}\right) \\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\right) - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left\lfloor \frac{j'-1}{2^{2i-1}} \right\rfloor\right) - \frac{j'}{2} \\
&= 2t_{n/2,1} + \frac{n}{2} - \frac{n}{2} \cdot 2^{-\lceil \ell/2 \rceil} \\
&\quad - \frac{1}{4}\left(2^{\lceil \ell/2 \rceil} - 2\right) - \left(t_{n/2,j'} - \frac{j'-1}{2} - t_{n/2,1}\right) - \frac{j'}{2} && \text{(by I.H.)} \\
&= 3t_{n/2,1} + \frac{n}{2} - n \cdot 2^{-\lceil \ell/2 \rceil - 1} - 2^{\lceil \ell/2 \rceil - 2} - t_{n/2,j'} \\
&= 2t_{n/2,1} + \frac{n}{2} - n \cdot 2^{-\lceil \ell/2 \rceil - 1} - t_{n/2,j'} && \text{(since } 2^{\lceil \ell/2 \rceil - 2} = t_{n/2,1}) \\
&= 2t_{n/2,1} + \frac{n}{2} - 2^{\lceil (\ell+1)/2 \rceil - 2} - t_{n/2,j'}. && \text{(since } n = 2^{\ell}) \qquad \text{(EC.10)}
\end{aligned}
$$

Recall we intend to show (EC.8), or equivalently, subtracting the LHS from the RHS yields a nonnegative value. The difference between the RHS expression in (EC.10) and the LHS expression in (EC.9) is

$$
\begin{aligned}
&\text{RHS} - \text{LHS} \\
&= \left(2t_{n/2,1} + \frac{n}{2} - 2^{\lceil (\ell+1)/2 \rceil - 2} - t_{n/2,j'}\right) \\
&\quad - \left(2t_{n/2,1} + n - 2^{\lceil (\ell+1)/2 \rceil - 1} - t_{n/2,k} - t_{n/2,k'}\right) \\
&= -\frac{n}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} + t_{n/2,k} + t_{n/2,k'} - t_{n/2,j'} \\
&= -\frac{n}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} + t_{n/2,1} \\
&\quad + (t_{n/2,k} - t_{n/2,1}) + (t_{n/2,k'} - t_{n/2,1}) - (t_{n/2,j'} - t_{n/2,1}) \\
&= -\frac{n}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} + t_{n/2,1} \\
&\quad + \sum_{i=0}^{\infty} \left(2^{i-2}\left(\left\lfloor \frac{k-1}{2^{2i-1}} \right\rfloor + \left\lfloor \frac{k'-1}{2^{2i-1}} \right\rfloor - \left\lfloor \frac{j'-1}{2^{2i-1}} \right\rfloor\right)\right) && \text{(writing thresholds as summations)} \\
&= -\frac{n}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} + t_{n/2,1} + \frac{k-1}{2} + \frac{k'-1}{2} - \frac{j'-1}{2} \\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lceil \frac{k}{2^{2i-1}} \right\rceil - 1 + \left\lceil \frac{k'}{2^{2i-1}} \right\rceil - 1 - \left\lceil \frac{j'}{2^{2i-1}} \right\rceil + 1\right)\right) && \text{(by Remark EC.1)} \\
&= -\frac{n}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} + t_{n/2,1} \\
&\quad + \frac{k-1}{2} + \frac{k'-1}{2} - \frac{j'-1}{2} - \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\right) && \text{(pulling constants from summation)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lceil\frac{k}{2^{2i-1}}\right\rceil + \left\lceil\frac{k'}{2^{2i-1}}\right\rceil - \left\lceil\frac{j'}{2^{2i-1}}\right\rceil\right)\right)\\
&\geq -\frac{n}{2} + 2^{\lceil(\ell+1)/2\rceil-2} + t_{n/2,1}\\
&\quad + \frac{k-1}{2} + \frac{k'-1}{2} - \frac{j'-1}{2} - \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\right)\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lceil\frac{k+k'}{2^{2i-1}}\right\rceil - \left\lceil\frac{j'}{2^{2i-1}}\right\rceil\right)\right) && (\text{since } \lceil x\rceil + \lceil y\rceil \geq \lceil x+y\rceil)\\
&= -\frac{n}{2} + 2^{\lceil(\ell+1)/2\rceil-2} + t_{n/2,1} - \frac{1}{2} + \frac{n/2}{2} - \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\right)\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\left\lceil\frac{j'+n/2}{2^{2i-1}}\right\rceil - \left\lceil\frac{j'}{2^{2i-1}}\right\rceil\right)\right) && (\text{since } k+k'=j=j'+n/2)\\
&= -\frac{n}{4} + 2^{\lceil(\ell+1)/2\rceil-2} + t_{n/2,1} - \frac{1}{2} - \frac{1}{4}\left(2^{\lceil\ell/2\rceil}-2\right)\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil} \left(2^{i-2}\left(\frac{n/2}{2^{2i-1}}\right)\right) && (\text{since } 2^{2i-1} \mid n/2)\\
&= -\frac{n}{4} + 2^{\lceil(\ell+1)/2\rceil-2} + t_{n/2,1} - 2^{\lceil\ell/2\rceil-2} + \frac{n}{2}\left(\frac{1}{2} - 2^{-\lceil\ell/2\rceil}\right)\\
&= 2^{\lceil(\ell+1)/2\rceil-2} + t_{n/2,1} - 2^{\lceil\ell/2\rceil-2} - 2^{\lceil(\ell+1)/2\rceil-2}\\
&= t_{n/2,1} - 2^{\lceil\ell/2\rceil-2}\\
&= 2^{\lceil(\ell-2)/2\rceil-1} - 2^{\lceil(\ell/2)\rceil-2}\\
&= 0.
\end{aligned}
$$

Hence (EC.8) holds for all $k \in K(j)$, as desired.

Case 3: If $1 \leq j \leq n/2$, then $K(j) = \{0, 1, \ldots, j\}$ by definition. To prove j is an argmin in (2), it suffices to show for all $k \in K(j)$

$$
n - t_{n/2,n/2-j+1} - t_{n/2,n/2-0+1} \leq n - t_{n/2,n/2-k+1} - t_{n/2,n/2-(j-k)+1},
$$

or equivalently,

$$
t_{n/2,n/2-k+1} + t_{n/2,n/2-(j-k)+1} \leq n/2 + t_{n/2,n/2-j+1}. \tag{EC.11}
$$

If $k = 0$, then (EC.11) holds with equality. Suppose $k > 0$, and let $k' = j - k$. Then the LHS of (EC.11) simplifies to

$$
\begin{aligned}
&t_{n/2,n/2-k+1} + t_{n/2,n/2-k'+1}\\
&\quad = t_{n/2,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor\frac{n/2-k+1-1}{2^{2i-1}}\right\rfloor\right)\\
&\quad\quad + t_{n/2,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor\frac{n/2-k'+1-1}{2^{2i-1}}\right\rfloor\right) && (\text{by I.H.})
\end{aligned}
$$

$$
\begin{aligned}
&= 2t_{n/2,1} + \frac{n/2-k}{2} + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{n/2-k}{2^{2i-1}}\right\rfloor\right)\\
&\quad + \frac{n/2-k'}{2} + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{n/2-k'}{2^{2i-1}}\right\rfloor\right)\\
&= 2t_{n/2,1} + \frac{n-k-k'}{2}\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lceil\frac{n/2-k+1}{2^{2i-1}}\right\rceil - 1\right)\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lceil\frac{n/2-k'+1}{2^{2i-1}}\right\rceil - 1\right) && \text{(by Remark EC.1)}\\
&= 2t_{n/2,1} + \frac{n-k-k'}{2} - 2\cdot\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\right)\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left(\frac{n/2}{2^{2i-1}} - \left\lfloor\frac{k-1}{2^{2i-1}}\right\rfloor\right)\right)\\
&\quad + \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left(\frac{n/2}{2^{2i-1}} - \left\lfloor\frac{k'-1}{2^{2i-1}}\right\rfloor\right)\right) && \text{(since } 2^{2i-1} \mid n/2)\\
&= 2t_{n/2,1} + \frac{n-k-k'}{2} - 2\cdot\left(\frac{1}{4}\left(2^{\lceil\ell/2\rceil}-2\right)\right)\\
&\quad + 2\cdot\frac{n}{2}\cdot\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{-i-1}\right)\\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{k-1}{2^{2i-1}}\right\rfloor\right)\\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{k'-1}{2^{2i-1}}\right\rfloor\right)\\
&= 2t_{n/2,1} + \frac{n}{2} - 2^{\lceil\ell/2\rceil-1}\\
&\quad + n\left(\frac{1}{2} - 2^{-\lceil\ell/2\rceil}\right)\\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{k-1}{2^{2i-1}}\right\rfloor\right) - \frac{k-1}{2}\\
&\quad - \sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{k'-1}{2^{2i-1}}\right\rfloor\right) - \frac{k'-1}{2}\\
&= n - n\cdot 2^{-\lceil\ell/2\rceil} && \text{(since } 2^{\lceil\ell/2\rceil-1} = 2t_{n/2,1})\\
&\quad - (t_{n/2,k} - t_{n/2,1}) - (t_{n/2,k'} - t_{n/2,1}) && \text{(by I.H.)}\\
&= n - 2^{\lceil(\ell+1)/2\rceil-1} - t_{n/2,k} - t_{n/2,k'} + 2t_{n/2,1}. && \text{(since } n = 2^{\ell}) \qquad \text{(EC.12)}
\end{aligned}
$$

Similarly, the RHS of (EC.11) simplifies to

$$
\begin{aligned}
&\frac{n}{2}+t_{n/2,n/2-j+1} \\
&\quad=\frac{n}{2}+t_{n/2,1}+\sum_{i=0}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{n/2-j}{2^{2i-1}}\right\rfloor\right) && \text{(by I.H.)}\\
&\quad=\frac{n}{2}+t_{n/2,1}+\frac{n/2-j}{2}+\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{n/2-j}{2^{2i-1}}\right\rfloor\right)\\
&\quad=\frac{n}{2}+t_{n/2,1}+\frac{n}{4}-\frac{j}{2}+\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left(\left\lceil\frac{n/2-j+1}{2^{2i-1}}\right\rceil-1\right)\right) && \text{(by Remark EC.1)}\\
&\quad=\frac{n}{2}+t_{n/2,1}+\frac{n}{4}-\frac{j}{2}-\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\right)\\
&\qquad+\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left(\frac{n/2}{2^{2i-1}}-\left\lfloor\frac{j-1}{2^{2i-1}}\right\rfloor\right)\right) && \text{(since } 2^{2i-1}\mid n/2)\\
&\quad=\frac{n}{2}+t_{n/2,1}+\frac{n}{4}-\frac{j}{2}-\frac{1}{4}\left(2^{\lceil\ell/2\rceil}-2\right)\\
&\qquad+\frac{n}{2}\cdot\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{-i-1}\right)-\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left\lfloor\frac{j-1}{2^{2i-1}}\right\rfloor\right)\\
&\quad=\frac{n}{2}+t_{n/2,1}+\frac{n}{4}-\frac{j-1}{2}-2^{\lceil\ell/2\rceil-2}\\
&\qquad+\frac{n}{2}\left(\frac{1}{2}-2^{-\lceil\ell/2\rceil}\right)-\left(t_{n/2,j}-\frac{j-1}{2}-t_{n/2,1}\right) && \text{(by I.H.)}\\
&\quad=\frac{n}{2}+2t_{n/2,1}+\frac{n}{4}-2^{\lceil\ell/2\rceil-2}\\
&\qquad+\frac{n}{4}-n\cdot 2^{-\lceil\ell/2\rceil-1}-t_{n/2,j}\\
&\quad=n+t_{n/2,1}-2^{\lceil(\ell+1)/2\rceil-2}-t_{n/2,j}. && \text{(since } 2^{\lceil\ell/2\rceil-2}=t_{n/2,1}) \qquad \text{(EC.13)}
\end{aligned}
$$

Recall we intend to show (EC.11), or equivalently, subtracting the LHS from the RHS yields a nonnegative value. The difference between the RHS expression in (EC.13) and the LHS expression in (EC.12) is

$$
\begin{aligned}
&\text{RHS}-\text{LHS}\\
&\quad=\left(n+t_{n/2,1}-2^{\lceil(\ell+1)/2\rceil-2}-t_{n/2,j}\right)\\
&\qquad-\left(n-2^{\lceil(\ell+1)/2\rceil-1}-t_{n/2,k}-t_{n/2,k'}+2t_{n/2,1}\right)\\
&\quad=t_{n/2,k}+t_{n/2,k'}-t_{n/2,j}-t_{n/2,1}+2^{\lceil(\ell+1)/2\rceil-2}\\
&\quad=(t_{n/2,k}-t_{n/2,1})+(t_{n/2,k'}-t_{n/2,1})\\
&\qquad-(t_{n/2,j}-t_{n/2,1})+2^{\lceil(\ell+1)/2\rceil-2}\\
&\quad=\sum_{i=0}^{\infty}\left(2^{i-2}\left(\left\lfloor\frac{k-1}{2^{2i-1}}\right\rfloor+\left\lfloor\frac{k'-1}{2^{2i-1}}\right\rfloor-\left\lfloor\frac{j-1}{2^{2i-1}}\right\rfloor\right)\right) && \text{(by I.H.)}\\
&\qquad+2^{\lceil(\ell+1)/2\rceil-2}\\
&\quad=\sum_{i=1}^{\lceil(\ell-2)/2\rceil}\left(2^{i-2}\left(\left\lfloor\frac{k-1}{2^{2i-1}}\right\rfloor+\left\lfloor\frac{k'-1}{2^{2i-1}}\right\rfloor-\left\lfloor\frac{j-1}{2^{2i-1}}\right\rfloor\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{k+k'-j-1}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} \\
&= \sum_{i=1}^{\lceil (\ell-2)/2 \rceil} \left(2^{i-2} \left(\left\lceil \frac{k}{2^{2i-1}} \right\rceil - 1 + \left\lceil \frac{k'}{2^{2i-1}} \right\rceil - 1 - \left\lceil \frac{j}{2^{2i-1}} \right\rceil + 1 \right) \right) && \text{(by Remark EC.1)} \\
&\quad + \frac{k+k'-j-1}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} \\
&\geq \sum_{i=1}^{\lceil (\ell-2)/2 \rceil} \left(2^{i-2} \left(\left\lceil \frac{k+k'}{2^{2i-1}} \right\rceil - \left\lceil \frac{j}{2^{2i-1}} \right\rceil \right) \right) && \text{(since } \lceil x \rceil + \lceil y \rceil \geq \lceil x+y \rceil\text{)} \\
&\quad - \sum_{i=1}^{\lceil (\ell-2)/2 \rceil} \left(2^{i-2} \right) + \frac{k+k'-j-1}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} \\
&= \sum_{i=1}^{\lceil (\ell-2)/2 \rceil} \left(2^{i-2} \left(\left\lceil \frac{j}{2^{2i-1}} \right\rceil - \left\lceil \frac{j}{2^{2i-1}} \right\rceil \right) \right) \\
&\quad - \frac{1}{4} \left(2^{\lceil \ell/2 \rceil} - 2 \right) + \frac{j-j-1}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} && \text{(since } k' = j-k\text{)} \\
&= 0 - 2^{\lceil \ell/2 \rceil - 2} + \frac{1}{2} - \frac{1}{2} + 2^{\lceil (\ell+1)/2 \rceil - 2} \\
&= 2^{\lceil (\ell+1)/2 \rceil - 2} - 2^{\lceil \ell/2 \rceil - 2} \\
&\geq 2^{\lceil \ell/2 \rceil - 2} - 2^{\lceil \ell/2 \rceil - 2} \\
&= 0.
\end{aligned}
$$

Hence (EC.11) holds for all $k \in K(j)$, as desired.

The above exhaustive case analysis proves (2) for $n = 2^\ell$. We will now derive the closed-form expressions in (1) for $n = 2^\ell$ using (2) and Lemma 2. Substituting the optimal choices of k from (2) into the recurrence of Lemma 2 gives

$$
\begin{aligned}
t_{n,j} &= \begin{cases} n - t_{n/2,n/2-j+1} - t_{n/2,n/2-(j-j)+1} & \text{if } 1 \leq j \leq n/2 \\ n - t_{n/2,n/2-n/2+1} - t_{n/2,n/2-(j-n/2)+1} & \text{otherwise} \end{cases} \\
&= \begin{cases} n/2 - t_{n/2,n/2-j+1} & \text{if } 1 \leq j \leq n/2 \\ n - t_{n/2,1} - t_{n/2,n/2-(j-n/2)+1} & \text{otherwise,} \end{cases} \qquad \text{(EC.14)}
\end{aligned}
$$

where the simplification of the upper case follows from the definition of $t_{n/2,n/2+1}$ as $n/2$. The proof is separated into two ranges of j with subcases for each based on the parity of ℓ. First, suppose $1 \leq j \leq n/2$. Then by (EC.14) and the induction hypothesis,

$$
\begin{aligned}
t_{n,j} &= n/2 - t_{n/2,n/2-j+1} \\
&= \frac{n}{2} - t_{n/2,1} - \sum_{i=0}^{\infty} \left(2^{i-2} \left\lfloor \frac{n/2-j+1-1}{2^{2i-1}} \right\rfloor \right) \\
&= \frac{n}{2} - t_{n/2,1} - \sum_{i=0}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2} \left\lfloor \frac{n/2-j}{2^{2i-1}} \right\rfloor \right) \\
&= \frac{n}{2} - 2^{\lceil (\ell-2)/2 \rceil - 1} - \frac{n/2-j}{2} - \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2} \left(\left\lceil \frac{n/2-j+1}{2^{2i-1}} \right\rceil - 1 \right) \right) && \text{(by Remark EC.1)} \\
&= \frac{n}{2} - 2^{\lceil \ell/2 \rceil - 2} - \frac{n}{4} + \frac{j}{2} + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2} \right) - \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2} \left\lceil \frac{n/2-j+1}{2^{2i-1}} \right\rceil \right)
\end{aligned}
$$

$$= \frac{n}{4} - 2^{\lceil \ell/2 \rceil - 2} + \frac{j}{2} + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2}\right) - \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2}\left(\frac{n/2}{2^{2i-1}} - \left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)\right) \qquad (\text{since } 2^{2i-1} \mid n/2)$$

$$= \frac{n}{4} - 2^{\lceil \ell/2 \rceil - 2} + \frac{j}{2} + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2}\right) - \frac{n}{2}\sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{-i-1}\right) + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil} \left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right). \qquad \text{(EC.15)}$$

If ℓ is even, then (EC.15) becomes

$$t_{n,j} = \frac{n}{4} - \frac{\sqrt{n}}{4} + \frac{j}{2} + \frac{1}{2}(\sqrt{n} - 1) - \frac{n}{2}\left(\frac{1}{2} - \frac{1}{2\sqrt{n}}\right) + \sum_{i=1}^{\ell/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)$$

$$= \frac{\sqrt{n}}{2} + \frac{j-1}{2} + \sum_{i=1}^{\ell/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)$$

$$= t_{n,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right), \qquad \text{(EC.16)}$$

matching the desired closed-form expression. If ℓ is odd, then (EC.15) becomes

$$t_{n,j} = \frac{n}{4} - \frac{\sqrt{2n}}{4} + \frac{j}{2} + \frac{1}{4}(\sqrt{2n} - 2) - \frac{n}{2}\left(\frac{1}{2} - \frac{1}{\sqrt{2n}}\right) + \sum_{i=1}^{(\ell-1)/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)$$

$$= \frac{\sqrt{2n}}{4} + \frac{j-1}{2} + \sum_{i=1}^{(\ell-1)/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)$$

$$= \frac{\sqrt{2n}}{4} + \sum_{i=0}^{(\ell-1)/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)$$

$$= t_{n,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right),$$

again matching the desired closed-form expression.

Now consider the remaining range, $n/2 < j \le n$. By (EC.14) and the induction hypothesis,

$$t_{n,j} = n - t_{n/2,1} - t_{n/2,n/2-(j-n/2)+1}$$

$$= n - t_{n/2,1} - \left(t_{n/2,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{n/2-(j-n/2)+1-1}{2^{2i-1}} \right\rfloor\right)\right)$$

$$= n - 2 \cdot 2^{\lceil (\ell-2)/2 \rceil - 1} - \sum_{i=0}^{\lceil (\ell-1)/2 \rceil}\left(2^{i-2}\left\lfloor \frac{n-j}{2^{2i-1}} \right\rfloor\right) \qquad \text{(by Lemma 1)}$$

$$= n - 2^{\lceil \ell/2 \rceil - 1} - \frac{n-j}{2} - \sum_{i=1}^{\lceil (\ell-1)/2 \rceil}\left(2^{i-2}\left(\left\lceil \frac{n-j+1}{2^{2i-1}} \right\rceil - 1\right)\right) \qquad \text{(by Remark EC.1)}$$

$$= \frac{n}{2} - 2^{\lceil \ell/2 \rceil - 1} + \frac{j}{2} + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil}\left(2^{i-2}\right) - \sum_{i=1}^{\lceil (\ell-1)/2 \rceil}\left(2^{i-2}\left(\frac{n}{2^{2i-1}} - \left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)\right) \qquad \text{(by Remark EC.1)}$$

$$= \frac{n}{2} - 2^{\lceil \ell/2 \rceil - 1} + \frac{j}{2} + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil}\left(2^{i-2}\right) - n \cdot \sum_{i=1}^{\lceil (\ell-1)/2 \rceil}\left(2^{-i-1}\right) + \sum_{i=1}^{\lceil (\ell-1)/2 \rceil}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right). \qquad \text{(EC.17)}$$

If ℓ is even, then (EC.17) becomes

$$t_{n,j} = \frac{n}{2} - \frac{\sqrt{n}}{2} + \frac{j}{2} + \sum_{i=1}^{\ell/2}\left(2^{i-2}\right) - n \cdot \sum_{i=1}^{\ell/2}\left(2^{-i-1}\right) + \sum_{i=1}^{\ell/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}} \right\rfloor\right)$$

$$
\begin{aligned}
&= \frac{n}{2} - \frac{\sqrt{n}}{2} + \frac{j}{2} + \frac{1}{2}(\sqrt{n}-1) - n\left(\frac{1}{2} - \frac{1}{2\sqrt{n}}\right) + \sum_{i=1}^{\ell/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \\
&= \frac{\sqrt{n}}{2} + \frac{j-1}{2} + \sum_{i=1}^{\ell/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \\
&= \frac{\sqrt{n}}{2} + \sum_{i=0}^{\ell/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \\
&= t_{n,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right),
\end{aligned}
$$

matching the desired closed-form expression. If ℓ is odd, then (EC.17) becomes

$$
\begin{aligned}
t_{n,j} &= \frac{n}{2} - 2^{(\ell+1)/2-1} + \frac{j}{2} + \sum_{i=1}^{(\ell-1)/2}\left(2^{i-2}\right) - n\cdot\sum_{i=1}^{(\ell-1)/2}\left(2^{-i-1}\right) + \sum_{i=1}^{(\ell-1)/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \\
&= \frac{n}{2} - \frac{\sqrt{2n}}{2} + \frac{j}{2} + \frac{1}{4}(\sqrt{2n}-2) - n\left(\frac{1}{2} - \frac{1}{\sqrt{2n}}\right) + \sum_{i=1}^{(\ell-1)/2-1}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \\
&= \frac{\sqrt{2n}}{4} + \frac{j-1}{2} + \sum_{i=1}^{(\ell-1)/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \\
&= \frac{\sqrt{2n}}{4} + \sum_{i=0}^{(\ell-1)/2}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) \qquad \text{(EC.18)} \\
&= t_{n,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right), \qquad \text{(by Lemma 1)}
\end{aligned}
$$

again matching the desired closed-form expression. The above cases are exhaustive, so (1) holds for all $1 \le j \le n$. By induction, (1) holds for all $r \in \mathbb{N}$, $n = 2^r$. Since the choices in (2) are optimal for all powers of two, and the number of districts in any subgame is a power of two when the starting number of districts is a power of two, the choices of (2) form a subgame perfect Nash equilibrium. □

THEOREM 1. *Let $f\colon [0,1] \to [0,1]$ be given by $f(v) = v$. Then $(f_{2^r})_{r=1}^{\infty}$ converges uniformly to f.*

Proof of Theorem 1. Since f_n is a step function, $|f_n(v) - f(v)|$ is maximized at some threshold point $\left(\frac{t_{n,j}}{n}, \frac{j-1}{n}\right)$ or $\left(\frac{t_{n,j}}{n}, \frac{j}{n}\right)$ for some $j \in [n] \equiv \{1, 2, \dots, n\}$. There are finitely many ($2n$) such points, so the maximum deviation of f_n from f is well-defined as

$$
\Delta_n = \sup_{j\in[n]}\left\{\max\left\{\frac{|t_{n,j}-(j-1)|}{n}, \frac{|t_{n,j}-j|}{n}\right\}\right\}.
$$

It suffices to show that for arbitrary $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all powers of two $n \ge N$, $\Delta_n \le \epsilon$.

An upper bound on $t_{n,j} - j$ is obtained by removing the floor from (1) and summing to infinity:

$$
\begin{aligned}
t_{n,j} - j &= t_{n,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) - j \\
&\le t_{n,1} + \sum_{i=0}^{\infty}\left(2^{i-2}\cdot 2^{1-2i}(j-1)\right) - j \\
&= 2^{\lceil(\log_2(n)-1)/2\rceil-1} + \frac{1}{2}(j-1)\left(\sum_{i=0}^{\infty}2^{-i}\right) - j
\end{aligned}
$$

$$= 2^{\lceil (\log_2(n)-1)/2\rceil -1} + \frac{1}{2}(j-1)\cdot 2 - j$$
$$\leq 2^{\lceil (\log_2(n)-1)/2\rceil -1},$$

which is either $\sqrt{n}/2$ or $\sqrt{2n}/4$ depending on the parity of $\log_2(n)$. Hence the upper bound is

$$t_{n,j} - j \leq \frac{\sqrt{n}}{2}. \tag{EC.19}$$

Similarly, a lower bound on $t_{n,j} - j$ is obtained by applying Lemma EC.1 to convert the floor to a ceiling and then removing the ceiling:

$$\begin{aligned}
t_{n,j} - j &= t_{n,1} + \sum_{i=0}^{\lceil \log_2(n)/2\rceil} \left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) - j \\
&= t_{n,1} + \frac{j-1}{2} + \sum_{i=1}^{\lceil \log_2(n)/2\rceil} \left(2^{i-2}\left\lfloor \frac{j-1}{2^{2i-1}}\right\rfloor\right) - j \\
&= t_{n,1} - \frac{j}{2} - \frac{1}{2} + \sum_{i=1}^{\lceil \log_2(n)/2\rceil} \left(2^{i-2}\left(\left\lceil \frac{j-1+1}{2^{2i-1}}\right\rceil - 1\right)\right) && \text{(by Remark EC.1)} \\
&\geq t_{n,1} - \frac{j}{2} - \frac{1}{2} - \sum_{i=1}^{\lceil \log_2(n)/2\rceil} \left(2^{i-2}\right) + \sum_{i=1}^{\lceil \log_2(n)/2\rceil} \left(\frac{j}{2^{i+1}}\right) \\
&= t_{n,1} - \frac{j}{2} - \frac{1}{2} - \frac{1}{2}\left(2^{\lceil \log_2(n)/2\rceil} - 1\right) + j\cdot \sum_{i=1}^{\lceil \log_2(n)/2\rceil} \left(2^{-i-1}\right) \\
&= t_{n,1} - \frac{j}{2} - 2^{\lceil \log_2(n)/2\rceil - 1} + j\left(\frac{1}{2}\left(1 - 2^{-\lceil \log_2(n)/2\rceil}\right)\right) \\
&= t_{n,1} - 2^{\lceil \log_2(n)/2\rceil - 1} - j\cdot 2^{-\lceil \log_2(n)/2\rceil - 1} \\
&\geq t_{n,1} - 2^{\lceil \log_2(n)/2\rceil - 1} - n\cdot 2^{-\lceil \log_2(n)/2\rceil - 1}. && \text{(since } j \leq n\text{)}
\end{aligned} \tag{EC.20}$$

Let $r \in \mathbb{N}$, $r = \log_2(n)$. If r is even, then by Lemma 1, (EC.20) becomes

$$\begin{aligned}
t_{n,j} - j &\geq \frac{\sqrt{n}}{2} - \frac{\sqrt{n}}{2} - \frac{n}{2\sqrt{n}} \\
&= -\frac{\sqrt{n}}{2}.
\end{aligned}$$

If r is odd, then by Lemma 1, (EC.20) becomes

$$\begin{aligned}
t_{n,j} - j &\geq \frac{\sqrt{2n}}{4} - \frac{\sqrt{2n}}{2} - \frac{n}{2\sqrt{2n}} \\
&= -\frac{\sqrt{2n}}{2}.
\end{aligned}$$

Hence the lower bound is

$$t_{n,j} - j \geq -\frac{\sqrt{2n}}{2}. \tag{EC.21}$$

The bounds (EC.19) and (EC.21) imply

$$\begin{aligned}
-\frac{1}{n}\cdot\frac{\sqrt{2n}}{2} &\leq \frac{t_{n,j} - j}{n} \leq \frac{1}{n}\cdot\frac{\sqrt{n}}{2} \\
-\frac{1}{\sqrt{2n}} &\leq \frac{t_{n,j} - j}{n} \leq \frac{1}{2\sqrt{n}}.
\end{aligned}$$

The (negative) lower bound is larger in magnitude, so

$$\frac{|t_{n,j} - j|}{n} \le \frac{1}{\sqrt{2n}}. \tag{EC.22}$$

Applying the above approach to obtain similar bounds for $t_{n,j} - (j-1)$ yields

$$\frac{|t_{n,j} - (j-1)|}{n} \le \frac{\sqrt{2n}}{2}. \tag{EC.23}$$

By (EC.22) and (EC.23),

$$\begin{aligned} \Delta_n &= \sup_{j\in[n]} \left\{ \max \left\{ \frac{|t_{n,j} - (j-1)|}{n}, \frac{|t_{n,j} - j|}{n} \right\} \right\} \\ &\le \frac{1}{\sqrt{2n}}. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary, and let $N = \lceil 1/(2\epsilon^2) \rceil$. Then for all $n \ge N$,

$$\begin{aligned} \Delta_n &\le \frac{1}{\sqrt{2n}} \\ &\le \frac{1}{\sqrt{2N}} \\ &\le \frac{1}{\sqrt{2\cdot 1/(2\epsilon^2)}} \\ &= \epsilon, \end{aligned}$$

as desired. Therefore, $(f_{2^r})_{r=1}^{\infty}$ converges uniformly to f. □

THEOREM 2. *Let $\delta \in (0, 1/2)$, $\gamma = 1/2 - \delta$, and $\alpha \in [\gamma, 1-\gamma]$. In the bisection protocol with packing parameter δ and n a power of two, the seat-share won by player 1 with a normalized vote-share of α obeys the limiting distribution*

$$\lim_{r\to\infty} f_{2^r}(\alpha) = \frac{\alpha - \gamma}{1 - 2\gamma}.$$

In the I-cut-you-freeze protocol with packing parameter δ and arbitrary n, the seat-share won by player 1 with a normalized vote-share of α obeys the limiting distribution

$$\lim_{n\to\infty} \frac{\sigma(n, \alpha n)}{n} = \begin{cases} 2\left(\frac{\alpha-\gamma}{1-2\gamma}\right)^2 & \text{for } \gamma \le \alpha \le \frac{1}{2} \\ 1 - 2\left(1 - \left(\frac{\alpha-\gamma}{1-2\gamma}\right)\right)^2 & \text{for } \frac{1}{2} < \alpha \le 1-\gamma, \end{cases}$$

where $\sigma(n, s)$ denotes the number of seats won by player 1 under optimum play of the I-cut-you-freeze protocol with n districts and player 1 vote-share s.

Proof of Theorem 2. Note that both protocols are feasible if and only if $\gamma \le \alpha \le 1-\gamma$. Requiring $s_{1,i} \in [\gamma, 1-\gamma]$ ensures that each player has a vote-share of at least γ in every district. If these vote-shares are preassigned, then the redistricting problem reduces to choosing n districts of size $1-2\gamma$ with the vote-shares s_1 and s_2 each reduced by γn. If player 1 begins with vote-share αn, then player 1 has $(\alpha-\gamma)n$ vote-share remaining after this preassignment. In the reduced problem, player 1's vote-share in each district can range from 0 to $1-2\gamma$, so the normalized vote-share is effectively $\alpha' = (\alpha-\gamma)/(1-2\gamma)$.

The limiting seats-votes curve for the bisection protocol follows from substituting α' for v in the RHS of the definition of f in Theorem 1. Similarly, the limiting seats-votes curve for the I-cut-you-freeze protocol follows from substituting α' for α in Theorem 2.3 of Pegden et al. (2017). □

THEOREM 3. *Let $G = (V, E)$ be an undirected graph with vertex weights $w : V \to \mathbb{R}_{\geq 0}$ and vertex populations $p : V \to \mathbb{R}_{\geq 0}$. For a vertex subset S, define $w(S) = \sum_{v \in S} w(v)$ and $p(S) = \sum_{v \in S} p(v)$. Let* BALANCEDBISECTION *be the problem of deciding whether G admits a bisection (connected 2-partition) into components D_1, D_2 such that $w(D_1) = w(D_2)$ and $p(D_1) = p(D_2)$. The problem* BALANCEDBISECTION *for $3 \times M$ grid graphs is NP-complete.*

Proof of Theorem 3. Clearly, BALANCEDBISECTION is in NP, since a satisfying bisection can be verified in polynomial time by summing the weights and populations for the two components, comparing the totals, and verifying the two components form a connected 2-partition. The proof of NP-hardness is by reduction from PARTITION, the known NP-complete problem of deciding whether a multiset S of M integers admits a partition S_1, S_2 such that S_1 and S_2 have the same sums.

Let $S = \{s_1, s_2, \ldots, s_M\}$ be an arbitrary instance of PARTITION. Construct a $3 \times M$ grid graph G with row indices 1 to 3 and column indices 1 to M. For all $i \in [M]$, assign weights $w(2, i) = s_i$ and $w(1, i) = w(3, i) = 0$. Similarly, for all $i \in [M]$, assign populations $p(2, i) = s_i$ and $p(1, i) = p(3, i) = 1$.

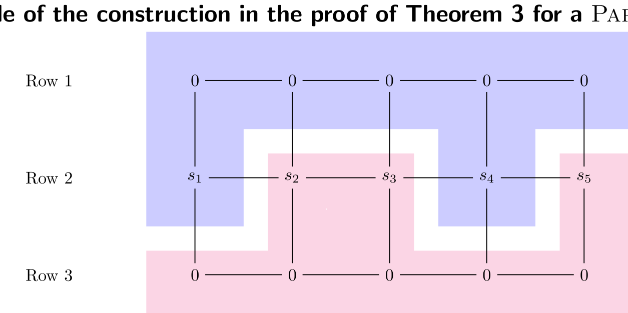
To prove correctness of the reduction, it suffices to show the following claim.

Claim: For any $\emptyset \neq S_1 \subset [M]$ with complement S_2, there is a connected 2-partition of G with $\{(2, i)\}_{i \in S_1}$ in one component and $\{(2, i)\}_{i \in S_2}$ in the other component.

Proof of Claim. Let R_1 be the first row of G, and let R_3 be the third row of G. Let one component be $R_1 \cup \{(2, i)\}_{i \in S_1}$, and let the other component be $R_3 \cup \{(2, i)\}_{i \in S_2}$. Then these two components comprise the desired 2-partition of G. □

Figure EC.12 illustrates the construction in the proof on a small example. A balanced bisection of the $3 \times M$ grid graph corresponds to a solution of the corresponding PARTITION instance.

Figure EC.12 **An example of the construction in the proof of Theorem 3 for a** PARTITION **instance with $M = 5$.**



PARTITION solution: $S_1 = \{s_2, s_3, s_5\}$, $S_2 = \{s_1, s_4\}$

Suppose the given PARTITION instance S has a solution S_1, S_2. Let G_1, G_2 be a bisection of G as guaranteed by the claim. Then G_1 and G_2 have the same total weights, as

$$\begin{aligned} w(G_1) &= \sum_{i \in S_1} w(2, i) \\ &= \sum_{i \in S_1} s_i \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i \in S_2} s_i \\
&= \sum_{i \in S_2} w(2,i) \\
&= w(G_2).
\end{aligned}
$$

Similarly, G_1 and G_2 have the same populations, as

$$
\begin{aligned}
p(G_1) &= M + \sum_{i \in S_1} p(2,i) \\
&= M + \sum_{i \in S_1} s_i \\
&= M + \sum_{i \in S_2} s_i \\
&= M + \sum_{i \in S_2} p(2,i) \\
&= p(G_2).
\end{aligned}
$$

Hence the given Partition instance has a solution if and only if the constructed BalancedBisection instance has a solution, so the reduction is proved. Since BalancedBisection is both NP-hard and in NP, we conclude it is NP-complete. □